Noise-induced transitions in spatially distributed and coupled pendulums

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We observe two classes of nonequilibrium noise-induced transitions in two-dimensional nearest-neighbor coupled pendulums. One is characterized by the drastic change of the shape of the stationary probability density and the other by the symmetry-breaking order parameter. The corresponding mechanisms are proposed. The nature of each transition is discussed. The cooperative effect of the two phenomena is presented. The mean-field results are consistent with the ones from the numerical calculations. [S1063-651X(97)02211-3]

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I. INTRODUCTION

One of the most actively studied nonequilibrium phenomena in systems coupled to fluctuating environment is the noise-induced transition that is characterized by the drastic change of the shape of the stationary probability density (SPD) of the corresponding Fokker-Planck equation [1]. Being analogous to the nonequilibrium phase transition in the deterministic system that occurs when the potential changes qualitatively, the transition in the stochastic system has been claimed in [1] to be most naturally characterized by the extrema of the corresponding SPD. At a critical noise intensity the number or the location of the extrema of the SPD changes, which represents the qualitative change of the macroscopic dynamics of the system (we denote this class of noise-induced transition as NIT1). In addition to low-dimensional dynamical systems, a globally coupled system has been shown to exhibit the peak splitting of the SPD leading to the NIT1 [2]. The other class of noise-induced transition is the one characterized by the symmetry-breaking order parameter, which resembles the conventional order-disorder equilibrium phase transition in its nature (this class will be denoted as NIT2). This phenomenon has been studied recently in a nearest-neighbor coupled system [3], not in a form displaying the NIT1 of the single-site system.

The NIT1 and the NIT2 have been studied independently with a lack of understanding the difference in their origins. The cooperative effects of these phenomena have never been studied either, to our knowledge. In this paper we observe these two classes of noise-induced nonequilibrium transitions and their cooperative phenomena in a spatially distributed and coupled system and compare the corresponding mechanisms. The single-site oscillator model with higher-harmonic pinning force for zero intrinsic frequency has been shown recently to exhibit both the NIT1 and the NIT2 [4]. We work in the nearest-neighbor coupled pendulums where the NIT1 and the NIT2 coexist. We clarify the ingredients leading to these phenomena in the coupled system: NIT1 is shown to be based on the peak splitting of the SPD of each uncoupled element and its interaction with the attractive coupling, while the NIT2 is on the local instability of each uncoupled element and its interaction with the coupling. The latter supports a general argument on the mechanism for the NIT2 in [5]. The cooperation of the NIT1 and the NIT2 is shown to result in the formation of a symmetry-breaking ordered phase with the configurations of double-peak SPD's. The NIT2 observed in this system shows a reentrant nature. In this paper, we work in the Stratonovich interpretation of stochastic processes. The results in the Ito version are also mentioned, supporting the arguments on the mechanisms for the transitions in the Stratonovich processes. We present a drastic difference generated by the two versions of the stochastic processes in the macroscopic behavior of a spatially distributed and coupled system.

In the next section the dynamics of the single-site system is derived. The single peak of the SPD at the deterministic stable fixed point is split into the ones at the deterministic stable fixed point and unstable fixed point above a critical noise intensity. We show the local instability around the deterministic fixed point in the Stratonovich interpretaion, but not in the Ito interpretation, which makes a drastic difference in the macroscopic dynamics of the coupled system. The system is studied in the mean-field approximation in Sec. III. The phase diagram is presented. The mean-field analysis predicts the results from the numerical calculations. We present the numerical results in Sec. IV. A summary and discussion follow in Sec. V.

II. ONE-SITE SYSTEM

We describe an overdamped single pendulum subject to a fluctuating field and thermal noise, which is given by [6,7]

$$\frac{d\phi}{dt} = -[b+\sigma\eta(t)]\sin\phi + \sigma_A\epsilon(t), \tag{1}$$

where the domain of ϕ is restricted to $[0,2\pi)$ imposing a periodic boundary condition and σ and σ_A measure the intensity of the multiplicative and additive noises, respectively. In the deterministic case ($\sigma=\sigma_A=0$) the system has a stable and an unstable fixed point at 0 and π, respectively. Equation (1) is also interpreted as an equation of motion describing various phenomena such as the synchronous oscillations in the visual cortex [8], Cooper pair tunneling in the Josephson junction [9], and the dynamics of sliding charge-density waves [10]. $\eta(t)$ and $\epsilon(t)$ are uncorrelated Gaussian white noises characterized by

$$\langle \eta(t)\rangle = \langle \epsilon(t)\rangle = 0, \tag{2}$$

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$$\langle \eta(t)\eta(t')\rangle = \langle \epsilon(t)\epsilon(t')\rangle = \delta(t-t').$$

Equation (1) possesses the reflection symmetry $\phi \rightarrow 2\pi - \phi$. Intuitively, one expects an instability of ϕ on average. It can be seen by the following equation of motion of the first moment [5]

$$\left\langle \frac{d\phi}{dt} \right\rangle = -\left(b - \frac{\sigma^2}{2} \right) \langle \phi \rangle, \tag{3}$$

where $\langle\ \rangle$ denotes the average with respect to the probability density. The probability density at $t=0$ is assumed to be a δ function. In Eq. (3) the higher order and the fluctuations around $\langle\phi\rangle$ are neglected. One can read an instability of the moment occurring at $\sigma^2 = \sigma_S^2 \equiv 2b$. Therefore, at short times, ϕ on average drifts out of 0. Equation (1) also shows an interesting long-time behavior. Depending on the sign of $b+\sigma\eta(t)$ when $\sigma_A = 0$, $\phi = 0$ and π can be either a stable fixed point or an unstable fixed point of Eq. (1). Therefore, the pendulum fluctuates spending most of its time at 0 and π. This leads to the peak splitting of the SPD at $\sigma^2 = 2b$, which can be read from Eq. (5) when $K=0$. It should be noted that the instability and the peak splitting in Eq. (1) corresponding to the short- and long: time behavior, respectively, are not related to each other even though the corresponding critical noise intensitites coincide. This can be seen also by a comparison with the results from the Ito process, where the peak splitting arises without the instability. In the Ito version, without the drift term by noise in Eq. (3) there is no instability at any parameter values. However, the peak splitting occurs at $\sigma^2 = b$.

III. MEAN-FIELD ANALYSIS OF THE COUPLED SYSTEM

Now we perform the mean-field analysis of the nearest-neighbor coupled pendulum of Eq. (1), which is given by

$$\frac{d\phi_i}{dt} = -[b+\sigma\eta_i(t)]\sin\phi_i - \frac{K}{d}\sum_{j,NN} \sin(\phi_i - \phi_j) + \sigma_A \epsilon_i(t), \tag{4}$$

where the sum runs over the d nearest neighbors of site i. We assume that the noises at one site characterized by Eq. (2) are not correlated with others. $K>0$ (attractive coupling) is assumed throughout this paper.

The above stochastic differential equation in the Stratonovich interpretation can be equivalently expressed as the Fokker-Planck equation

$$\begin{aligned}\frac{\partial P(\{\phi_i\},t)}{\partial t} = \sum_i \frac{\partial}{\partial \phi_i}\Bigg[\Bigg(b\ \sin\phi_i - \frac{\sigma^2}{2}\sin\phi_i\cos\phi_i \\ + K(\sin\phi_i)C(t) - (\cos\phi_i)S(t)\Bigg) P(\{\phi_i\},t)\Bigg] \\ + \sum_i \frac{\partial^2}{\partial {\phi_i}^2}\left[\left(\frac{\sigma_A^2}{2} + \frac{\sigma^2}{2}\sin^2\phi_i\right) P(\{\phi_i\},t)\right],\end{aligned} \tag{5}$$

where $C(t)$ and $S(t)$ are defined as $C(t) = (1/d)\,\Sigma_{j,NN}\cos\phi_j(t)$ and $S(t) = (1/d)\Sigma_{j,NN}\sin\phi_j(t)$, respectively. In the mean-field approximation the steady-state equation (5) can be written as a single variable

$$\frac{dP(\phi)}{d\phi} + \frac{P(\phi)[b\ \sin\phi + (\sigma^2/2)\sin\phi\ \cos\phi + K(\sin\phi)C - K(\cos\phi)S] + J}{(\sigma^2/2)\ \sin^2\phi + \sigma_A^2/2} = 0, \tag{6}$$

where J is the constant probability current that is imposed by the boundary condition $P(\phi+2\pi) = P(\phi)$. C and S are the steady-state values of the averages of $\cos\phi_j$ and $\sin\phi_j$, respectively. With the translational invariance and the isotropy of the steady state of the system, C and S satisfy the self-consistent equations

$$C = \int_0^{2\pi} d\phi(\cos\phi)P(\phi),$$

$$S = \int_0^{2\pi} d\phi(\sin\phi)P(\phi). \tag{7}$$

The stationary solution of the Fokker-Planck equation in the mean-field approximation is given by

$$P(\phi) = N_m \exp[-U(\phi)],$$

$$\begin{aligned}U(\phi) = \left(\frac{1}{2} - \frac{b+KC}{\sqrt{\sigma^4+\sigma^2\sigma_A^2}}\right)\ln\left(\sqrt{1+\frac{\sigma_A^2}{\sigma^2}} + \cos\phi\right) \\ + \left(\frac{1}{2} + \frac{b+KC}{\sqrt{\sigma^4+\sigma^2\sigma_A^2}}\right)\ln\left(\sqrt{1+\frac{\sigma_A^2}{\sigma^2}} - \cos\phi\right) \\ - \frac{2KS}{\sigma\sigma_A}\tan^{-1}\left(\frac{\sigma}{\sigma_A}\sin\phi\right),\end{aligned} \tag{8}$$

where N_m is the normalization factor.

Solving Eq. (7) with the stationary probability density in Eq. (8), we obtain the phase diagram of the system as in Fig. 1. The corresponding order parameters for the NIT1 and the NIT2 are the number of peaks of $P(\phi)$ and S, respectively. S_s is the reflection conserving phase whose configurations have SPD's with single peak at 0. In this phase $C=1$ and $S=0$. The B_s phase represents the reflection symmetry-breaking phase. The SPD's of the B_s phase configurations have a single peak located neither at $\phi=0$ nor at $\phi=\pi$ producing nonzero value of the order parameter S. B_d denotes the symmetry-breaking phase, where the SPD's have double peaks located in a reflection symmetry breaking fashion. In both the B_s and B_d phases $C\neq 1$ and $S\neq 0$. S_d is the symmetry conserving phase with SPD's having double peaks

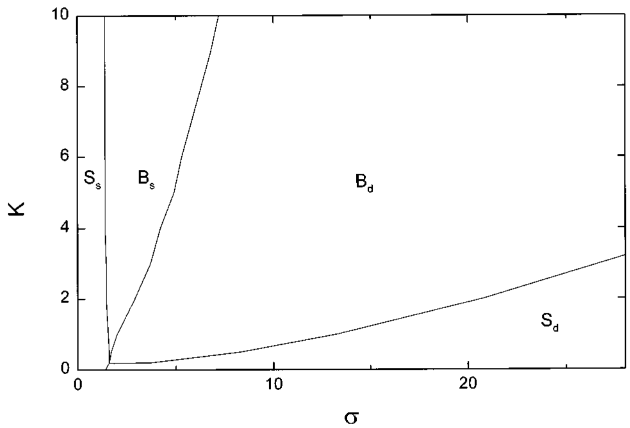

FIG. 1. Phase diagram for $b=1.0$ and $\sigma_A=0.03$ from the mean-field approximation.

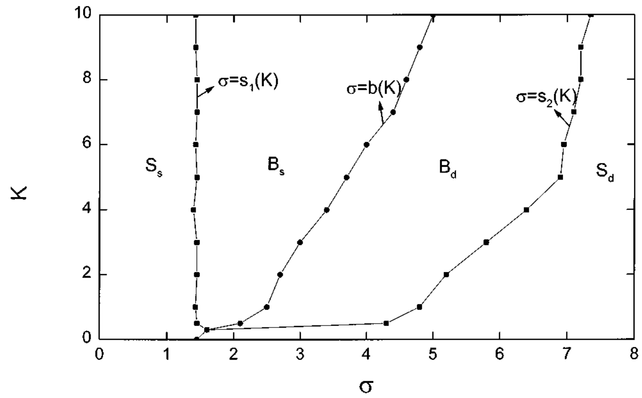

FIG. 2. Phase diagram for $b=1.0$ and $\sigma_A=0.03$ obtained from the numerical calculations of Eq. (4). Lines are a guide to the eye.

located exactly at $\phi=0$ and π. In the S_d phase $C\neq 1$ and $S=0$. The detailed nature of each phase will be discussed in the next section. The SPD's corresponding to these four phases have qualitatively same configurations as the ones in Fig. 3.

The Ito interpretation of Eq. (4) gives rise to the same SPD as the one in Eq. (8), but with $\frac{1}{2}$ replaced by 1. The numerical solutions of the coupled equations (7) and (8) show that the symmetry-breaking phase does not appear in the Ito version. The solutions of the self-consistent equations consist of the S_s or the S_d phase configurations in the Ito interpretation.

IV. NUMERICAL RESULTS

Now we present the numerical results of Eq. (4). We set $O=(1/N^2)\,\Sigma_i \sin\psi_i$ as an order parameter for the NIT2, which is given by

$$re^{i\psi_j}=\langle\cos\phi_j\rangle+i\langle\sin\phi_j\rangle, \tag{9}$$

where $N\times N$ is the system size, $r^2=\langle\cos\phi_i\rangle^2+\langle\sin\phi_i\rangle^2$, the subscript i denotes the site, and $\langle\ \rangle$ is the average over time. In the following simulations we have used the second-order Runge-Kutta method with discrete time steps of $\Delta t=0.01$ with random initial configurations. At each run, 3×10^5 time steps per site have been used to compute averages. The system size is 20×20.

Figure 2 shows the phase diagram of NIT1 and NIT2 for $b=1.0$ and $\sigma_A=0.03$. $\sigma=s_1(K)$ and $\sigma=s_2(K)$ in Fig. 2 denote the critical lines of the NIT2 and $\sigma=b(k)$ the one of the NIT1. We use the same notations as the ones in Sec. III to denote the four phases.

In Fig. 3 we present the SPD's corresponding to the S_s, B_s, B_d, and S_d phases, respectively, for $b=1.0$, $K=1.0$, and $\sigma_A=0.03$. The solid line in Fig. 3 shows that the pendulums are mostly at $\phi=0$ in the steady state when the noise is very weak. The $\sigma=s_1(K)$ line that separates the B_s phase from the S_s phase in Fig. 2 is independent of the coupling strength, except when the coupling is very weak. It coincides with $\sigma^2=\sigma_S^2$, where the instability of the moment of the single-site system occurs. This and the mean-field result in Fig. 1 prove that the local instability of each uncoupled element causes the NIT2 in the coupled system. When $s_1^2(K)<\sigma^2<b^2(K)$, the peak of the SPD is shifted from 0 being located neither at 0 nor at π as the $\sigma=2.0$ line in Fig. 2 shows. As the instability indicates, each element tends be kicked out of 0 on average. The pendulums showing the short-time instability attract the neighbored elements through the coupling to gather around some nonzero ϕ in the steady state. Thus the instability of the single-site system at short times interacting with the attractive coupling results in the shift of the peak of the SPD to ϕ that is different from 0 or π. The location of this shifted peak is determined by the strength of noise and coupling. The $\sigma=b(K)$ line is the critical line of the NIT1 representing the peak splitting of the SPD of the coupled system. The NIT1 of Eq. (4) originated from the peak splitting of the SPD of each uncoupled element and is not related with the instability of the moment of uncoupled element. This argument is consistent with the results in the Ito processes, where is the peak splitting of the SPD of each uncoupled element without the instability of the moment leads the coupled system to the NIT1. When $b^2(K)<\sigma^2<s_2^2(K)$, not only is the peak shifted as in the B_s phase due to the instability interacting with the coupling, but the peak of the SPD of the coupled system is split. This results in the double peak shifted from 0 and π in a localized form due to the attractive coupling, as the dashed line in Fig.

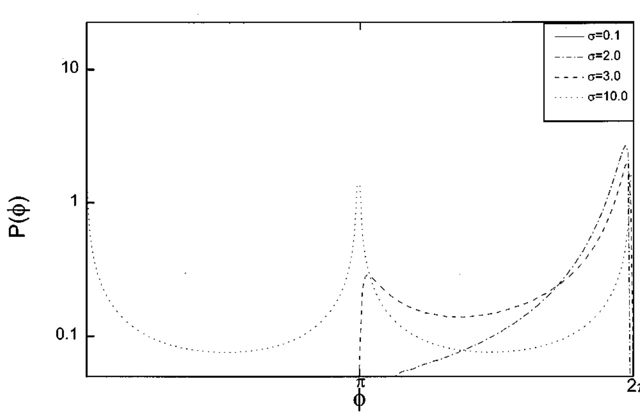

FIG. 3. Plot of the SPD when $K=1.0$ and $\sigma_A=0.03$ for some noise intensities. The pendulums for the $\sigma=2.0$ and 3.0 cases are localized at $[0,\pi]$ with probability P_0 or at $[\pi,2\pi]$ with probability $1-P_0$, depending on the initial conditions. This shows a bifurcation in the steady states.

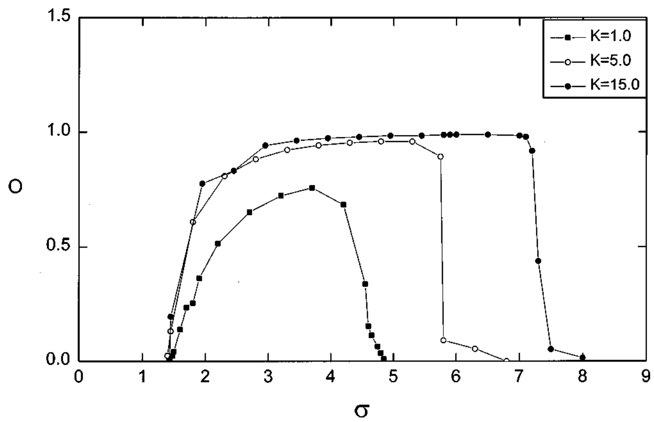

FIG. 4. For $b=1.0$ and $\sigma_A=0.03$, O's are plotted versus σ when $K=1.0$, 5.0, and 15.0. Lines are a guide to the eye. We present only the positive values of O neglecting the mirror image with respect to the x axis.

3 shows. The pendulums in the configuration corresponding to the dashed line in Fig. 3 are localized at $[0,\pi]$ or at $[\pi,2\pi]$ even though they are more broadly distributed than in the configurations of S_s and B_s phases so that the order parameter O is different from 0 representing a symmetry-breaking ordered phase. As the noise intensity increases, the disordering effect of the noise dominates the localization due to the coupling so that the pendulums get more broadly distributed. The two peaks approach 0 and π, respectively, as noise intensity increases until the symmetry-conserving S_d phase appears. The appearance of the B_d phase is a cooperative behavior of the instability and the peak splitting of the single-site system and the coupling. When $\sigma^2 > s_2^2(K)$, the noise intensity is strong enough to exhibit a disordering effect. This dominates the attractive coupling. Therefore, the localization of the pendulum populations due to the coupling is supressed so that the split peaks are located at 0 and π. The pendulums tend to be broadly distributed over the whole range of ϕ due to the disordering effect of multiplicative noise as the dotted line in Fig. 3. The broken reflection symmetry is therefore restored resulting in the reentrant behavior of the NIT2. The same transition pattern was observed for any coupling strength as for $K=1.0$.

In Fig. 4 the order parameter O is plotted versus σ for some fixed coupling strengths. As the coupling strength increases for a fixed noise intensity, the pendulums get more localized, resulting in the increase of the order-parameter value.

The Ito interpretation of stochastic processes exhibits a drastic difference in the macroscopic behavior of the system. The Ito version of Eq. (1) has no noise-dependent term in Eq. (3) and therefore generates no instability. However, the peak splitting of the SPD in the single-site system occurs. Numerical results present the NIT1 of Eq. (4), but not the NIT2 in the Ito version. With the results in the Stratonovich version, this strongly reflects that when interacting with the coupling, the peak splitting of the SPD in the single-site system results in the peak splitting of the SPD in the coupled system, while the local instability of the single-site system results in the symmetry-breaking phase transition.

V. SUMMARY AND DISCUSSION

In conclusion, we studied a nearest-neighbor coupled system where the two classes of noise-induced transition phenomena coexist. We found various phases resulting from the interaction of the short-time dynamics and the steady-state behavior of the single-site system with the coupling. The key ingredients for the results in this paper are the local instability induced by noise, peak splitting due to the existence of the unstable fixed point, and diffusive coupling. The mean-field results qualitatively fit the numerical ones. Comparing the results from the two interpretations of the stochastic processes with each other, we clarified the corresponding mechanisms for the NIT1 and the NIT2 in coupled system, which have been studied independently with a lack of understanding the difference in their origins. The two interpretations of stochastic processes were therefore shown to generate a drastic difference in the macroscopic behavior of the coupled system, which previously has been studied in the single-site system [11] or in the continuum limit of coupled system [12]. The existence of the various phases from the cooperative and competitive effects of the two transitions was confirmed by the mean-field results and the numerical results. The two results, however, show a relatively large quantitative discrepancy. The discrepancy in the locations of the phase transition points becomes larger as the coupling gets stronger. More detailed theoretical and numerical investigations are left for further study to understand the critical phenomena of the model.

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